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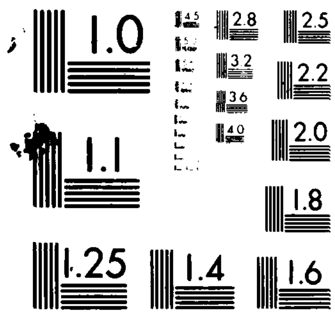

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AN ASYMPTOTIC STOCHASTIC VIEW OF ANTICIPATION IN A NOISY DUEL (I)*

Dan R. Royalty[†], J. Colby Kegley[††], H.T. David[††], and R.W. Berger[††]

Abstract. The noisy duel between two equally accurate duelists, possessing respectively 1 and 2 bullets, is viewed in the light of certain asymptotic distributions for their times of first fire. These distributions, reflecting the weaker player's need to anticipate, are derived from an approximating sequence of simultaneous games.

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The noisy duel between two equally accurate duelists, possessing respectively 1 and 2 bullets, is viewed in the light of certain asymptotic distributions for their times of first fire. These distributions, reflecting the weaker player's need to anticipate, are derived from an approximating sequence of simultaneous games.

1. Introduction. The classical noisy duel, say of 1 bullet vs 2, with equal accuracy and a continuum of opportunities for firing, does not have a saddle point. Remedies have included 1) "continuizing these ammunition stores ([12], [14]), 2) settling for only ε-good solutions ([2], [11]), and 3) placing certain extreme behavioral demands ([4]) on the weaker duelist, calling for anticipating the opponent's current action, i.e., for instantaneously adapting his own firing plans to that action. A fourth possible remedy, introduced in [15] and in a sense dual to the first, is to allow only a finite set of firing opportunities, thus creating approximating finite simultaneous games ([17]), which, as finite games of perfect recall ([13]), admit randomized behavioral solutions, i.e., solutions in the form of firing time distributions. The asymptotic versions of these distributions model the anticipation required of the weaker duelist in [4].

A not unimportant element of zero-sum two-person game theory is the quantification (typically in stochastic fashion) of compelling not otherwise quantified features of strategy; the quantification [19] of bluffing in poker for example, by behavioral randomization calling for "irrational" play now and then; or the quantification, by an atom of firing probability reserved for the bitter end, of the Kamikaze commitment of a value - less single - salvo fighter in silent duel with a continuously firing opponent. Still another example, then, is our modeling of the weaker duelist's instantaneous anticipation by the fact that his asymptotic first-firing distribution has infinite expectation and straddles that of his opponent. It seems fair to say that this modeling will be of interest to the degree that its details will not be predictable intuitively a priori, and indeed they do not seem to be. It does seem impossible to foretell, starting only from knowing that the players' good strategies somehow are to be asymptotically stochastically portrayed, whether the same normalization will work for both players, where the two asymptotic firing time distributions will be situated with respect to one another, or whether I's need to anticipate will be reflected in incomplete convergence in law of his normalized firing time, beyond the infinite expectation of its asymptotic distribution. What does turn out is given below in (1). Whether portions of (1) are extrapolatable from $\Gamma(1,2)$ to games of timing generally, we do not as yet know, although we do know that they are so extrapolatable to $\Gamma(m,n)$, as outlined in Section 6.

Consider then two equally accurate duelists, players I and II, possessing respectively 1 and 2 bullets, facing a finite grid, with uniform mesh ε, of joint firing opportunities, with the points of the grid calibrated from 0 to 1 according to bullet lethality p. Dynamic programming yields an optimal behavioral solution for each duelist, which, among other things, specifies optimal probabilities, *greater than zero and less than one*, of first firing (strictly speaking just "firing" in the case of I) at certain contiguous grid points near $p = 1/3$. These probabilities are conditional on neither duelist having yet fired, and may, for each duelist, be interpreted as determining an optimal unconditional distribution of time of (first) fire near $p = 1/3$, say $\Phi_{1,\varepsilon}$ for player I and $\Phi_{2,\varepsilon}$ for player II. Players I and II are to independently select the time of their first shot, respectively according to $\Phi_{1,\varepsilon}$ and $\Phi_{2,\varepsilon}$, and to fire at the time selected unless the opponent has fired before that time.

Though both $\Phi_{1,\varepsilon}$ and $\Phi_{2,\varepsilon}$ converge to degeneracy at $p = 1/3$ with increasing numbers of firing opportunities, the microscopy of location-scale norming (the same norming for $\Phi_{1,\varepsilon}$ and $\Phi_{2,\varepsilon}$) reveals two asymptotic distributions on $[-1,+\infty)$, say Φ_1 and Φ_2, with densities

$f_1(t)$ and $f_2(t)$, such that (cf. fig. 1)

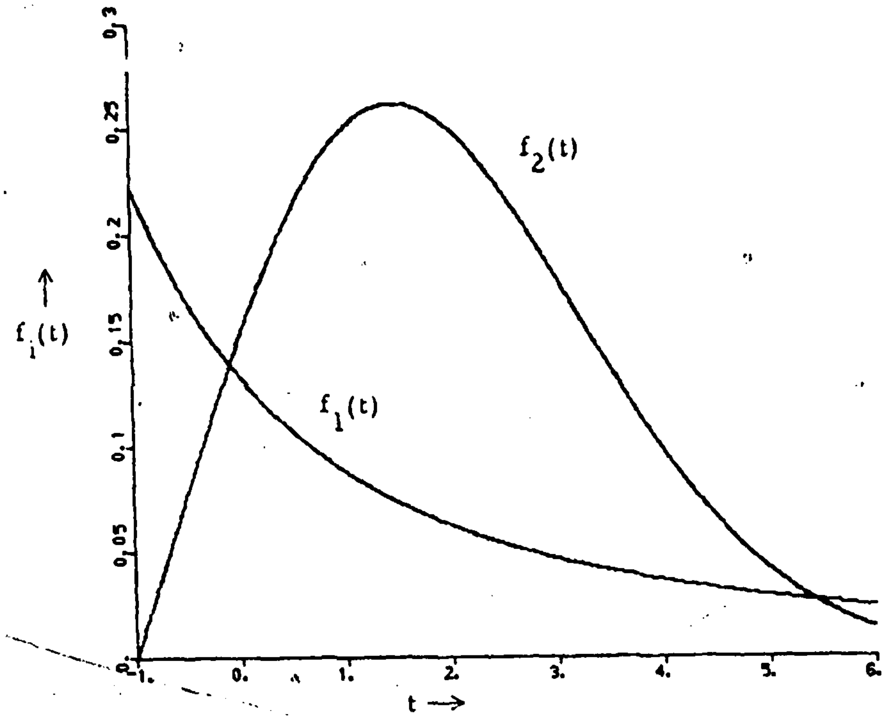

Figure 1. Asymptotic densities of times of (first) fire.

(1a) $$\Phi_1(t) \gtrless \Phi_2(t) \quad \text{for} \quad t \lessgtr 1 ,$$

(1b) $$\Phi_1(+\infty) = 1 ,$$

and

(1c) $$\text{mode } (\Phi_1) < \text{mode } (\Phi_2), \text{ mean } (\Phi_2) < \text{mean } (\Phi_1) ,$$

with

(1d) $$\text{mean } (\Phi_1) = +\infty .$$

As indicated in Section 5, relation (1a) may be derived with the aid of certain relatively weak hypotheses clearly indicated by extensive computer runs; relations (1b), (1c) and (1d) follow from certain strengthened versions of these hypotheses, allowing the expression (34) of Φ_1 and (38) of Φ_2 in terms of certain Hermite equation solutions.

We submit that (1a) and (1c) model I's need to anticipate by I's asymptotic firing distribution "straddling" that of II, and that this need to anticipate is also modeled by the upper-tail mass of I's asymptotic firing distribution, mass enough to cause (1d) (though not enough to prevent (1b)).

The next 4 sections detail the origins of the findings (1) concerning Φ_1 and Φ_2, together with the derivation of a certain associated differential equation whose solution is discussed in [9]. Section 6 indicates the extension of the results to $\Gamma(m,n)$.

2. The dynamic programs. As pointed out in the introduction, restricting firing opportunities to a finite grid converts the noisy duel into a finite simultaneous game (cf. [17]). The successive information sets of this simultaneous game corresponding to no one having yet fired tell the story: They are the only non-trivial information sets for the duel $\Gamma(1,2)$ treated here. Since each player possesses precisely one such information set per grid point, we shall speak below of a player's behavior "at a grid point p", when we in fact mean to speak of his behavior at the information set corresponding to that grid point.

Assuming a uniform grid of mesh ϵ, at each of the successive grid points $p = 1, 1-\varepsilon, 1-2\varepsilon, \ldots, 0$ proceeding backwards from unity the players face a 2×2 game with matrix

(2a)

	F	$\bar{F}$
F	0	1
$\bar{F}$	-1	0

when $p = 1$, and

(2b)

	F	$\bar{F}$
F	$-(1-p)^2$	$2p-1$
$\bar{F}$	$-p$	$u[p+\varepsilon]$

when $p < 1$,

where $u[p+\varepsilon]$ is the value of the game at the grid point $p+\varepsilon$ (see [18]). Since $2p-1$ is never the minimum in its row for $p > 0$, the game has a saddle point at (F,F) as long as $p \geq p^*$, where $p^* \doteq 0.382$ is the solution of $1-p^* = \sqrt{p^*}$.

At a grid point $p < p^*$, one of three things can happen. There may be a saddle point at $(\bar{F},\bar{F})$, i.e., $-p \geq u[p+\varepsilon] \geq 2p-1$, which forces $p \leq 1/3$.

Also, there are grids of arbitrarily small mesh for which some grid point $p < p^*$ produces a saddle point at $(\bar{F},F)$. For example, if $\delta > 0$ and N is a positive integer such that

$$2N-1 > \max\{2(1-p^*)/\delta,\ (1-p^*)/(1-2p^*)\},$$

then the grid of mesh $\varepsilon = 2(1-p^*)/(2N-1)$, which is less than δ, produces a $(\bar{F},F)$ saddle point when $p = 1-N\varepsilon = p^*-\varepsilon/2$. It is shown following (4) that, for given ε, a $(\bar{F},F)$ saddle point can occur at most one grid point p.

The third possibility for a grid point $p < p^*$ is that the corresponding game has no pure saddle point. As is shown in [18], the value $u[p+\varepsilon]$ determines the mixed value $u[p]$ and the optimal conditional firing probabilities $x_1[p]$ for I and $x_2[p]$ for II:

$$u[p] = -\{u[p+\varepsilon](1-p)^2 + p(1-2p)\}/\{u[p+\varepsilon] + p(1-p)\}, \tag{3a}$$

$$x_1[p] = \{u[p+\varepsilon] + p\}/\{u[p+\varepsilon] + p(1-p)\}, \tag{3b}$$

$$x_2[p] = \{u[p+\varepsilon] + 1-2p\}/\{u[p+\varepsilon] + p(1-p)\}. \tag{3c}$$

By inspection of the matrix (2b) of the game, we have in turn that $-(1-p)^2 < -p$, i.e., $p^2 - 3p + 1 > 0$, that $u[p+\varepsilon] < -p$, and that $u[p+\varepsilon] < 2p-1$. The last inequality implies that the denominator in each of the formulas (3) is less than $-p^2 + 3p-1$, which is negative. Evidently, then, both $x_1[p]$ and $x_2[p]$ are positive. Moreover, it is straightforward to show that formulas (3) imply that

$$u[p] + p = -(p^2-3p+1)\, x_1[p], \tag{4a}$$

$$u[p] + 1-2p = -p^2\, x_2[p], \tag{4b}$$

$$u[p+\varepsilon] - u[p] = \{u[p+\varepsilon] + p(1-p)\}\, x_1[p]\, x_2[p], \tag{4c}$$

all of which are negative.

In particular, $u[p] < -p < -(p-\varepsilon)$, so the game at the next grid point $p-\varepsilon$ has no saddle point at $(\bar{F},F)$. This implies that if, in the backwards induction process, a $(\bar{F},F)$ saddle point occurs for the first time at a grid point $p+\varepsilon$, where $\varepsilon < (p^*-1/3)/2$, then necessarily $p^*-\varepsilon < p+\varepsilon \leq p^*$. Also, if this happens, then the game for the next grid point $\hat{p}$ has the matrix (2b) with $u[p+\varepsilon] = -(p+\varepsilon)$, so it does not have $(\overline{F},F)$ as a saddle point. This discussion shows that the $(\overline{F},F)$ saddle point phenomenon can occur at most once for a given mesh ε, either at p^* (if that happens to be a grid point) or at the first grid point less than p^*.

Hence, we can conclude that, for ε small, there is a largest grid point $\overline{p}(\varepsilon)$ at which the game has no pure saddle point, and that $p^* - 2\varepsilon < \overline{p}(\varepsilon) < p^*$. Starting at $\overline{p}(\varepsilon)$ and continuing backwards, we move through a succession of grid points p at which mixed optimal strategies obtain. Since (4c) implies that $u[p+\varepsilon] < u[p]$, the value of the game increases as we proceed backwards. Moreover, the game at $p = \overline{p}(\varepsilon) + \varepsilon$ has a pure saddle point at either (F,F) or $(\overline{F},F)$. So, the value $u[\overline{p}(\varepsilon) + \varepsilon]$ has one of the forms $-(1-p)^2$ or $-p$, and is therefore greater than -1. It follows that we must eventually arrive at a smallest grid point $\hat{p}(\varepsilon)$ for which $u[\hat{p}(\varepsilon) + \varepsilon] < 2\hat{p}(\varepsilon) - 1$, i.e., for which random strategies occur.

Finally, the matrix (2b) for the game at $p = \hat{p}(\varepsilon) - \varepsilon$ has $u[\hat{p}(\varepsilon)]$ in the $(\overline{F},\overline{F})$ entry, so it has a saddle point there. Proceeding inductively backwards, then, the same situation holds for $0 \leq p < \hat{p}(\varepsilon)$.

The conditional firing probabilities $x_1[\hat{p}(\varepsilon)],\ldots,x_1[\overline{p}(\varepsilon)]$ determine the unconditional firing time distribution $\Phi_{1,\varepsilon}$ of the introduction. An analogous interpretation holds as well for the sequence $x_2[\hat{p}(\varepsilon)],\ldots,x_2[\overline{p}(\varepsilon)]$ and the corresponding distribution $\Phi_{2,\varepsilon}$. The next sections discuss the scaling of $\Phi_{1,\varepsilon}$ and $\Phi_{2,\varepsilon}$, and consequent derivation of their asymptotic counterparts Φ_1 and Φ_2.

3. Asymptotic scaled values and probabilities of fire. For a finite grid $(1, 1-\varepsilon, 1-2\varepsilon, \ldots, 0)$, let $\hat{p}(\varepsilon)$ and $\overline{p}(\varepsilon)$, respectively, be the smallest and largest grid points from Section 2 at which the corresponding 2×2 game has no pure saddle point. Also, let $\beta(\varepsilon) = 1/3-\hat{p}(\varepsilon)$, let $\theta(\varepsilon) = \varepsilon/\beta(\varepsilon)$ and, for $t \geq -1$, let $p(t,\varepsilon)$ be the largest grid point that does not exceed $1/3 + t\beta(\varepsilon)$. This allows writing

$$p(-1,\varepsilon) = \hat{p}(\varepsilon) \tag{5a}$$

and, for $t > -1$,

$$p(t,\varepsilon) = 1/3 + t\beta(\varepsilon) + f(t,\varepsilon), \tag{5b}$$

where $-\varepsilon < f(t,\varepsilon) \leq 0$. We introduce the step-functions $u(t,\varepsilon) \equiv u[p(t,\varepsilon)]$, $x_1(t,\varepsilon) \equiv x_1[p(t,\varepsilon)]$, $x_2(t,\varepsilon) \equiv x_2[p(t,\varepsilon)]$, and note that the definitions of $p(t,\varepsilon)$ and $\theta(\varepsilon)$ imply that $u(t + \theta(\varepsilon),\varepsilon) = u[p(t,\varepsilon) + \varepsilon]$ and similarly for the other two functions evaluated at $(t + \theta(\varepsilon),\varepsilon)$.

Extensive computer implementations in [18] of the dynamic programs of Section 2 point to the following hypotheses concerning $\beta(\varepsilon)$ and $u(t,\varepsilon)$:

HYPOTHESIS A. There is a positive number C such that

$$\beta(\varepsilon) = C\sqrt{\varepsilon} + o(\sqrt{\varepsilon}). \tag{6a}$$

HYPOTHESIS B. There is a function $v(t)$, differentiable for $t > -1$ and right-differentiable at $t = -1$, such that, if $t \geq -1$, then

$$u(t,\varepsilon) = -1/3 - v(t)\theta(\varepsilon) + g(t,\varepsilon), \tag{6b}$$

where for each $t, g(t,\varepsilon) = o(\sqrt{\varepsilon})$ and $g(t+\theta(\varepsilon),\varepsilon) - g(t,\varepsilon) = o(\varepsilon)$.

A simple application of the definition of the derivative $v(t)$ results in

$$u(t+\theta(\varepsilon),\varepsilon) - u(t,\varepsilon) = -v'(t)\theta^2(\varepsilon) + h(t,\varepsilon), \tag{7}$$

where for each t, $h(t,\varepsilon) = o(\varepsilon)$. We proceed to use results from Section 2 to derive a differential equation that $v(t)$ must satisfy under Hypotheses A and B.

First, for $t \geq -1$, we can choose ε sufficiently small so that the 2×2 game for the grid point $p(t,\varepsilon)$ has no pure saddle point, i.e., so that $\hat{p}(\varepsilon) \leq p(t,\varepsilon) \leq \bar{p}(\varepsilon)$. This holds by definition when $t = -1$. For $t > -1$, the inequality $\hat{p}(\varepsilon) \leq 1/3 + t\beta(\varepsilon)$ is equivalent to $0 \leq (1+t)\beta(\varepsilon)/\sqrt{\varepsilon}$. But this must hold for ε small, since Hypothesis A implies that as $\varepsilon \to 0^+$, $(1+t)\beta(\varepsilon)/\sqrt{\varepsilon} \to (1+t)C$, which is positive. Also, $1/3 + t\beta(\varepsilon) \to 1/3$ while the inequalities $p^*-2\varepsilon < \bar{p}(\varepsilon) < p^*$ force $\bar{p}(\varepsilon) \to p^*$ as $\varepsilon \to 0^+$, so $1/3 + t\beta(\varepsilon) < \bar{p}(\varepsilon)$ for small ε.

Hence, we can apply (4c) to (7), obtaining

$$\begin{aligned} -v'(t) + \frac{h(t,\varepsilon)}{\theta^2(\varepsilon)} &= [u(t+\theta(\varepsilon),\varepsilon) - u(t,\varepsilon)]/\theta^2(\varepsilon) \\ &= \{u(t+\theta(\varepsilon),\varepsilon) + p(t,\varepsilon)(1-p(t,\varepsilon))\}\frac{x_1(t,\varepsilon)}{\theta(\varepsilon)}\,\frac{x_2(t,\varepsilon)}{\theta(\varepsilon)} \end{aligned} \tag{8}$$

The definition of $\theta(\varepsilon)$, together with Hypothesis A and the fact that $h(t,\varepsilon) = o(\varepsilon)$, imply that $h(t,\varepsilon)/\theta^2(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. And relations (5) and (6) imply that the expression in braces on the right side of (8) approaches $-1/9$ as $\varepsilon \to 0^+$.

Relations (4a) and (4b) can be used in conjunction with (6b) to calculate

(9) $$x_i(t) = \lim_{\varepsilon \to 0^+} x_i(t,\varepsilon)/\theta(\varepsilon) \quad \text{for} \quad i = 1,2.$$

For, if we use (4a) we can write

(10) $$\frac{x_1(t,\varepsilon)}{\theta(\varepsilon)} = -\left\{\frac{u(t,\varepsilon)+1/3}{\theta(\varepsilon)} + \frac{p(t,\varepsilon)-1/3}{\theta(\varepsilon)}\right\} \div [p^2(t,\varepsilon)-3p(t,\varepsilon)+1].$$

If we use relations (5) together with Hypotheses A and B, we obtain, as $\varepsilon \to 0^+$,

(11) $$x_1(t) = 9[v(t) - C^2 t].$$

Similarly, if we start from (4b), we get

(12) $$\frac{x_2(t,\varepsilon)}{\theta(\varepsilon)} = -\left\{\frac{u(t,\varepsilon)+1/3}{\theta(\varepsilon)} + \frac{2(1/3-p(t,\varepsilon))}{\theta(\varepsilon)}\right\} \div p^2(t,\varepsilon),$$

which, as $\varepsilon \to 0^+$, yields

(13) $$x_2(t) = 9[v(t) + 2C^2 t].$$

Returning to relation (8) with these results and letting $\varepsilon \to 0^+$, we find that for, $t \geq -1$, $v(t)$ must be a solution of the Riccati equation

(14) $$v'(t) = 1/9\ x_1(t)\ x_2(t) = 9[v(t)-C^2 t][v(t)+2C^2 t].$$

We can obtain an initial condition for $v(t)$ at $t = -1$ by examining (12) with $t = -1$ or, what amounts to the same thing, (4b) with $p = \hat{p}(\varepsilon)$. Noting that $\theta(\varepsilon) > 0$ for ε small (since $\theta(\varepsilon)/\sqrt{\varepsilon} \to 1/C$ as $\varepsilon \to 0^+$), we have from (4b) that

(15) $$0 > -\hat{p}^2(\varepsilon)x_2[\hat{p}(\varepsilon)]/\theta(\varepsilon) = (u[\hat{p}(\varepsilon)]+1-2\hat{p}(\varepsilon))/\theta(\varepsilon).$$

On the other hand, the definition of $\hat{p}(\varepsilon)$ and consideration of the 2×2 matrix (2b) for $p = \hat{p}(\varepsilon)-\varepsilon$ shows that $u[\hat{p}(\varepsilon)] \geq 2(\hat{p}(\varepsilon)-\varepsilon) - 1$. Hence, the right member of (15) is not less than $-2\varepsilon/\theta(\varepsilon) = -2\beta(\varepsilon)$, which approaches 0 as $\varepsilon \to 0^+$. Since $\hat{p}^2(\varepsilon) \to 1/9$ as $\varepsilon \to 0^+$, we conclude that

(16) $$x_2(-1) = \lim_{\varepsilon\to 0^+} x_2(-1,\varepsilon)/\theta(\varepsilon) = 0.$$

By (13), this can be written as

(17) $$v(-1) = 2C^2.$$

Furthermore, the limit relations (9) together with the negativity of the quantities in (4a) and (4b) imply that $x_1(t) \geq 0$ and $x_2(t) \geq 0$ $t \geq -1$. Hence, (14) implies $v'(t) \geq 0$. Applying this to (13) gives $x_2'(t) \geq 18C^2 > 0$. Hence, by (16),

(18) $$x_2(t) > 0 \quad \text{for} \quad t > -1.$$

To show that $x_1(t)$ is strictly positive, suppose $x_1(t_0) = 0$ for some t_0. Then (11) and (17) rule out $t_0 = -1$. But if this were to occur at some $t_0 > -1$, then the fact that $x_1(t) \geq 0$ for $t \geq -1$ would imply that $x_1(t)$ would have a global minimum at an interior point of its domain. So, $x_1'(t_0) = 0$ which, by (11), says $v'(t_0) = C^2$. But (14) with $x_1(t_0) = 0$ says $v'(t_0) = 0$, a contradiction. Therefore,

(19) $$x_1(t) > 0 \quad \text{for} \quad t \geq -1.$$

This, coupled with (11) and (12), gives

(20) $$x_2(t) > 27C^2t \quad \text{for} \quad t \geq -1,$$

which is a stronger statement than (18) for $t > 0$.

J. C. Kegley has proved in [9] that, given any $C > 0$, there is a unique solution of (14) that satisfies (19) for large values of t, and that the solution exists for all t. Two forms of the solution are given in terms of the confluent hypergeometric functions. One of the forms, which will be used in Section 5, is

(21) $$v(t) = C^2t - \frac{\delta}{9}\frac{\psi'(s)}{\psi(s)},$$

where $\delta = C\sqrt{27/2}$, $s = \delta t$, and $\psi(s)$ is the solution of the Hermite equation

(22a) $$\psi''(s) - 2s\,\psi'(s) - 2/3\,\psi(s) = 0$$

that satisfies the initial conditions

(22b) $$\psi(0) = 1, \ \psi'(0) = -2\Gamma(2/3)/\Gamma(1/6).$$

The second form of the solution is used to prove that condition (17) then determines C uniquely, namely, $C = -z_0/\sqrt{27}$, where z_0 is the zero of the Weber parabolic cylinder function $D_{2/3}(z)$. A numerical approximation of z_0 furnished by R. J. Lambert gives

(22c) $$C = 0.090637 \pm 3\times10^{-6}.$$

This is in good agreement with the value .09067 of $\beta(\varepsilon)/\sqrt{\varepsilon}$ obtained for $\varepsilon \doteq 1.5\times10^{-8}$, the smallest of the ε's underlying the computer runs leading to Hypotheses A and B, and enhances the credibility of these hypotheses.

4. Asymptotic distributions of first fire. Consider now, for $t \geq -1$ and ε small and positive, the probability $\overline{\Phi}_{1,\varepsilon}(1/3+\beta(\varepsilon)\cdot t) \equiv 1-\Phi_{1,\varepsilon}(1/3+\beta(\varepsilon)\cdot t)$ that player I fires at or after the grid point $p(t,\varepsilon) + \varepsilon$ given, as always, that player II has not already fired. If we denote by $I(t,\varepsilon)$ the collection of all grid points p with $\hat{p}(\varepsilon) \leq p \leq p(t,\varepsilon)$, where ε is small enough so that $\hat{p}(\varepsilon) \leq p(t,\varepsilon) \leq \overline{p}(\varepsilon)$, then we have, by independence of random moves under a behavioral strategy,

$$\overline{\Phi}_{1,\varepsilon}(1/3+\beta(\varepsilon)\cdot t) = \prod_{p\varepsilon I(t,\varepsilon)} (1-x_1[p]). \tag{23}$$

By analyzing the natural logarithm of $\overline{\Phi}_{1,\varepsilon}(1/3+\beta(\varepsilon)\cdot t)$, we will show that, as $\varepsilon \to 0^+$, $\overline{\Phi}_{1,\varepsilon}(1/3+\beta(\varepsilon)\cdot t)$ tends to the limit

$$\overline{\Phi}_1(t) \equiv \exp(-\textstyle\int_{-1}^{t} x_1(\tau)d\tau). \tag{24}$$

Thinking of $t \geq -1$ as fixed, recall now that, as indicated above following (5b), $x_1(\tau,\varepsilon)/\theta(\varepsilon)$ is a step-function for $-1 \leq \tau \leq t$, with uniform step-size $\theta(\varepsilon)$. Hence, we may write

$$\sum_{p\varepsilon I(t,\varepsilon)} x_1[p] = \int_{-1}^{t(\varepsilon)} [x_1(\tau,\varepsilon)/\theta(\varepsilon)]d\tau, \tag{25}$$

where $t(\varepsilon) \leq t < t(\varepsilon) + \theta(\varepsilon)$ by definition of $p(t,\varepsilon)$.

In order to show that, as $\varepsilon \to 0^+$, $x_1(\tau,\varepsilon)/\theta(\varepsilon)$ tends to $x_1(\tau)$ uniformly for $-1 \leq \tau \leq t$, we invoke Hypothesis B. This hypothesis says that the step-functions $[u(\tau,\varepsilon) + 1/3]/\theta(\varepsilon)$ converge pointwise to $-v(\tau)$, which is assumed to be differentiable and, hence, continuous. But (4c) and the fact that $\theta(\varepsilon) > 0$ shows that for each ε, the corresponding step-function is a non-increasing function of τ. By a well-known theorem (see for example page 90 of [3]), it follows that $[u(\tau,\varepsilon) + 1/3]/\theta(\varepsilon)$ converges uniformly to $-v(\tau)$ on the compact interval $[-1,t]$.

Inspection of relation (10) and the definition of $p(t,\varepsilon)$ now makes it clear that the step-functions $x_1(\tau,\varepsilon)/\theta(\varepsilon)$ converge uniformly to $x_1(\tau)$ on $[-1,t]$, which, since $t(\varepsilon) \to t$ as $\varepsilon \to 0^+$, allows concluding from (25) that

$$(26)\qquad \sum_{p\varepsilon I(t,\varepsilon)} x_1[p] \to \int_{-1}^{t} x_1(\tau)d\tau \quad \text{as} \quad \varepsilon \to 0^+ .$$

We also note that, since $\theta(\varepsilon) \to 0$ as $\varepsilon \to 0^+$, the uniform convergence of $x_1(\tau,\varepsilon)/\theta(\varepsilon)$ to $x_1(\tau)$ for $-1 \leq \tau \leq t$ forces

$$(27)\qquad \sup_{p\varepsilon I(t,\varepsilon)} x_1[p] \to 0 \quad \text{as} \quad \varepsilon \to 0^+.$$

Now we examine

$$(28)\qquad \log \overline{\Phi}_{1,\varepsilon}(1/3+\beta(\varepsilon)\cdot t) + \sum_{p\varepsilon I(t,\varepsilon)} x_1[p] = \sum_{p\varepsilon I(t,\varepsilon)} \{\log(1-x_1[p])+x_1[p]\}.$$

By Taylor's theorem, we have $\log(1-x_1[p]) + x_1[p] = r(p)$, where $|r(p)| \leq (x_1[p])^2/2(1-x_1[p])^2$. If we choose ε small enough so that, by (27), $\sup_{p\varepsilon I(t,\varepsilon)} x_1[p] \leq 1-1/\sqrt{2}$, we find that $|r(p)| \leq (x_1[p])^2$ for $p\varepsilon I(t,\varepsilon)$. Therefore the right member of (28) has absolute value not exceeding

$$\sum_{p\in I(t,\varepsilon)} (x_1[p])^2 \leq \sup_{p\in I(t,\varepsilon)} x_1[p] \cdot \sum_{p\in I(t,\varepsilon)} x_1[p],$$

which approaches 0 as $\varepsilon \to 0^+$ by (26) and (27). By (26), then,

$$\log \overline{\Phi}_{1,\varepsilon}(1/3+\beta(\varepsilon)\cdot t) \to -\int_{-1}^{t} x_1(\tau)d\tau \quad \text{as} \quad \varepsilon \to 0^+,$$

so $\overline{\Phi}_{1,\varepsilon}(1/3+\beta(\varepsilon)\cdot t)$ converges to $\overline{\Phi}_1(t)$ as defined by (24).

Hence, $\Phi_{1,\varepsilon}(1/3+\beta(\varepsilon)\cdot t) \equiv 1 - \overline{\Phi}_{1,\varepsilon}(1/3+\beta(\varepsilon)\cdot t)$ tends, as $\varepsilon \to 0^+$, to a cdf $\Phi_1(t) \equiv 1 - \overline{\Phi}_1(t)$ with hazard rate $x_1(t)$ (see for example [1] for discussions of the concept of hazard rate in its usual context of reliability theory), which we call I's asymptotic distribution of time of (first) fire. Analogously, $x_2(t)$ is the hazard rate of II's asymptotic distribution $\Phi_2(t)$ of time of first fire.

Hence, if $T_{1,\varepsilon}$ is Player I's time of (first) fire under mesh ε, $(T_{1,\varepsilon}-1/3)/\beta(\varepsilon)$ is seen to converge in law to the cdf Φ_1 with hazard rate x_1, and an entirely analogous argument and conclusion hold for Player II. Normalization is by the location parameter 1/3 and the scale parameter $\beta(\varepsilon)$, the latter in effect stretching the shrinking interval $[\hat{p}(\varepsilon), 1/3]$ into the unit interval $[-1,0]$. Of course, in view of Theorem 2 (page 42) of [7], essentially no other normalization will work.

Having obtained these results, we can now look back on their derivations and see why Hypotheses A and B constitute an efficient set of assumptions. We do this by attempting to alter Hypotheses A and B to the natural alternative Hypotheses A', B' and D below, equally plausible in the light of the computer runs, and showing that we are driven back to A and B.

To begin with, we now postulate, with $\theta(\varepsilon) = \varepsilon/\beta(\varepsilon)$ and $\beta(\varepsilon) = 1/3 - \hat{p}(\varepsilon)$,

HYPOTHESIS A'. $\beta(\varepsilon) \to 0^+$ as $\varepsilon \to 0^+$, which implies that $p(t,\varepsilon) \to 1/3$,

HYPOTHESIS B'. There exists a function $v(t)$ on $[-1,+\infty)$ such that $(u(t,\varepsilon) + 1/3)/\theta(\varepsilon)$ has limit $-v(t)$ as $\varepsilon \to 0^+$,

and

HYPOTHESIS D. There exists a function $x_1(t)$ on $[-1,+\infty)$ such that $x_1(t,\varepsilon)/\theta(\varepsilon)$ has limit $x_1(t)$ as $\varepsilon \to 0^+$.

Then (10) implies the existence of the limit, as $\varepsilon \to 0^+$, of $(p(t,\varepsilon) - 1/3)/\theta(\varepsilon)$ on $[-1,+\infty)$, and the definition of $p(t,\varepsilon)$ then implies the existence of the limit, say C, of $\beta(\varepsilon)/\sqrt{\varepsilon}$ as $\varepsilon \to 0^+$, which in turn implies, by relation (12), the existence of the limit $x_2(t)$ of $x_2(t,\varepsilon)/\theta(\varepsilon)$ on $[-1,+\infty)$ as $\varepsilon \to 0^+$.

Now, if it were true that $C = 0$, then (11) and (13) would say $x_1(t) = 9v(t) = x_2(t)$ for all $t \geq -1$. Furthermore, we could arrive at the key relation (8) with the left member replaced by

$$-[v(t+\theta(\varepsilon)) - v(t)]/\theta(\varepsilon) + [g(t+\theta(\varepsilon),\varepsilon) - g(t,\varepsilon)]/\theta^2(\varepsilon),$$

and there seems to be no reasonable way to utilize this relation without strengthening Hypothesis B' to Hypothesis B, in which case the differential equation (14) becomes $v'(t) = 9v^2(t)$, while the initial condition (17) becomes $v(-1) = 0$. But the only solution of this initial value problem is $v(t) \equiv 0$. That in turn forces $x_1(t) \equiv 0$, corresponding to no non-degenerate cdf's $\Phi_i(t)$. This we are forced to conclude that $C > 0$, which amounts to strengthening Hypothesis A' to Hypothesis A.

5. Comparison of asymptotic distributions. Some relationships between the cdf's $\Phi_i(t)$ can be derived under hypotheses weaker than A and B. If we assume, again based on the pre-asymptotic computer runs, that A holds but that B is weakened to:

B': For $t \geq -1$, $\lim\limits_{\varepsilon \to 0^+} \dfrac{u(t,\varepsilon)+1/3}{\theta(\varepsilon)} = -\,v(t)$ exists,

then our derivations in Section 3 obtain the existence and non-negativity of the limits $x_i(t) = \lim\limits_{\varepsilon\to 0^+} x_i(t,\varepsilon)/\theta(\varepsilon)$, and also formulas (11) and (13) for $x_i(t)$. From these formulas we can see that the hazard rates are related by

$$x_2(t) = x_1(t) + 27C^2 t \ . \tag{29}$$

In the derivation of the formulas

$$\overline{\Phi}_i(t) = \exp\left(-\int_{-1}^{t} x_i(\tau)\,d\tau\right) \tag{30}$$

we needed A and

B": Hypothesis B' holds and $v(t)$ is continuous.

Thus, under A and B", relations (29) and (30) give

$$\overline{\Phi}_2(t) = \exp[\delta^2(1-t^2)]\ \overline{\Phi}_1(t), \tag{31}$$

where $\delta = C\sqrt{27/2}$. This implies the comparisons

(32) $\Phi_1(t) > \Phi_2(t)$ for $-1 < t < 1$ and $\Phi_1(t) < \Phi_2(t)$ for $t > 1$,

which is to say that Φ_2 is more highly concentrated about the normalized firing time 1 than is Φ_1.

Hypotheses A and B'' also imply that the densities

(33) $$f_i(t) \equiv -\Phi_i'(t) = x_i(t)\exp(-\int_{-1}^{t} x_i(\tau)d\tau)$$

are defined and non-negative. However, it is not apparent that rigorous analysis of these densities can be done without invoking the full force of A and B. Since these hypotheses eventually lead to formula (21) for v(t), it is evident that the analysis of the densities depends critically on the properties of the solutions $\psi(s)$ of problem (22). For example, formulas (11) and (21) imply that $x_1(t) = -\frac{d}{dt}\ln|\psi(s)|$, hence that

(34) $$\Phi_1(t) = \psi(s)/\psi(-\delta).$$

As is shown in [9], $\psi(s) > 0$ for all s and $\psi(s) \to 0^+$ as $s = \delta t \to +\infty$, so that $\int_{-1}^{\infty} x_1(t)dt = +\infty$, and, in view of (33),

(35) $$\int_{-1}^{\infty} f_1(t)dt = 1;$$

this is to say that, even though I's asymptotic firing time distribution is heavy-tailed compared to that of II, the convergence of I's normalized firing time distribution is complete.

Relation (29) implies that $\int_{-1}^{\infty} x_2(t)dt = +\infty$ also. Hence (33) yields as well that $\int_{-1}^{\infty} f_2(t)dt = 1$.

Inspection of figure 1, based on the computer runs, suggests the results that

(36) (i) mode $(\Phi_1) = -1$ and (ii) mode $(\Phi_2) > 0$.

If both A and B hold, then the results of [9] can be used to establish (i), based on (34) and the facts that $\psi(s)$ is positive, decreasing, and convex, and also to derive (ii) from a special relationship between $\psi(s)$ and the Weber parabolic cylinder functions $D_{-1/3}$ and $D_{2/3}$. In particular, the differential equations analysis leads to an approximation mode $(\Phi_2) \doteq 1.5058$, furnished by R. J. Lambert, which, as figure 1 suggests, is in accordance with the computer runs.

No amount of inspection, however, will suggest that

(37) (i) mean $(\Phi_2) < \infty$ and (ii) mean $(\Phi_1) = \infty$.

Under A and B, we can use (31), (34), and the definition of $\overline{\Phi}_1(t)$ to get

(38) $$\overline{\Phi}_2(t) = \exp[\delta^2(1-t^2)]\ \psi(s)/\psi(-\delta).$$

The facts that ψ is positive and decreasing force $\overline{\Phi}_2(t)$ to be trapped between 0 and $\exp[\delta^2(1-t^2)]$, which leads to (37(i)). The infinite expectation for player I is established in [9], based on the fact that $\overline{\Phi}_1(t) = \psi(\delta t)/\psi(-\delta)$ behaves like $t^{-1/3}$ as $t \to \infty$.

6. The m vs. n case. We conclude with a brief outline of how results analogous to the above are obtained for the m vs. n equal-accuracy noisy duel, where $m < n$. The details of the derivations

can be found in [9] and [10].

The analysis begins with the replacement of the matrix (2b) by

$$\begin{array}{ccc} & F & \overline{F} \\ F & (1-p)^2 u_{m-1,n-1}[p+\varepsilon] & p + (1-p)u_{m-1,n}[p+\varepsilon] \\ \overline{F} & -p + (1-p)u_{m,n-1}[p+\varepsilon] & u_{m,n}[p+\varepsilon] \end{array}$$

where $u_{i,j}[p+\varepsilon]$ is the value of the i vs. j game at the grid point $p+\varepsilon$. In particular, this value is 0 if $i = j$ and is -1 if $i = 0$.

Hypotheses A and B are modified suitably, again based on computer runs for some small values of m and n. Based on those hypotheses, we can adapt the methods of Sections 3 and 4 above to derive the following formulas for the asymptotic distributions Φ_1 and Φ_2 of the players' normalized-times of first fire.

The cdf $\overline{\Phi}_1(t) = \psi(s)/\psi(s_o)$, where $s = \delta(t+\eta)$ with $\delta > 0$, $-\eta$ is the value of $t > -1$ where the two players' hazard rates are equal (which is zero in the case 1 vs. 2), $s_o = \delta(-1+\eta)$, and ψ is the solution of the Hermite equation

$$\psi''(s) - 2s\psi'(s) - 4a\psi(s) = 0, \; a = m/2(m+n),$$

that satisfies the initial conditions

$$\psi(0) = 1, \; \psi'(0) = -2\Gamma(a+\tfrac{1}{2})/\Gamma(a).$$

The cdf $\overline{\Phi}_2(t) = G(t)\,\overline{\Phi}_1(t)$, where $G(t) = \exp(s_o^2 - s^2)$.

Based on these formulas and the properties of the function ψ derived in [9], we find that

$$\Phi_1(t) > \Phi_2(t) \quad \text{for} \quad -1 < t < 1-2\eta, \quad \Phi_1(t) < \Phi_2(t) \quad \text{for} \quad t > 1-2\eta$$

$$\lim_{t\to\infty} \Phi_1(t) = 1$$

and that mode $(\Phi_1) = -1 < -\eta <$ mode (Φ_2) while $-1 <$ mean $(\Phi_2) <$ mean (Φ_1), with mean $(\Phi_1) = +\infty$. Thus, we have the *inequalities* (1a) and (1c) for the m vs. n case while maintaining the weaker player's complete convergence of normalized firing time, and infinite normalized firing time expectation.

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